The Linear Vector Operator of Quaternions.

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This paper has for its object the development of the algebra of the linear vector operator, entirely from a quaternion point of view, which amounts to an extension or development of nonions. It is presumed that so much of the theory of the linear vector operator is known as is developed in works on Quaternions, as Tait, 3d edition, or Hamilton's Elements. I consider first the expression of such an operator, as ϕ, in terms of three numbers, a, b, c, which depend only on the three (latent) roots of ϕ, and a unit operator ι, which depends only on the axes of ϕ. I then consider ϕ as dependent on nine operators which are linearly independent, each of nullity two, three of vacuity two, and six of vacuity three. (For definitions of latent roots, axes, nullity, and vacuity, see Taber, "Theory of Matrices," Amer. Jour. Math., vol. 12, pp. 355 and 362.)

I.

1. The function ϕ may be written in the form

$$\phi = a + b\iota + c\iota^2, \tag{1}$$

a, b, c being scalars and ι a linear vector operator such that $\iota^3.\rho = \rho$ or $\iota^3 = 1$. The operator ι involves implicitly six scalars, since its equation $\iota^3 - 1 = 0$ determines explicitly three of the usual nine scalars involved in every general linear vector function. The proof of the existence of this form of ϕ will consist simply in determining explicitly each element of it. The three roots and three axes of ϕ are supposed known.

We have at once

$$\phi^2 = a^2 + 2bc + (c^2 + 2ab)\,\iota + (b^2 + 2ac)\,\iota^2, \tag{2}$$
$$\phi^3 = a^3 + b^3 + c^3 + 6abc + 3\,(ac^2 + ba^2 + cb^2)\,\iota + 3\,(a^2c + b^2a + c^2b)\,\iota^2. \tag{3}$$

Whence it is easy to deduce the equation

$$\phi^3 - 3a\,.\,\phi^2 + 3\,(a^2 - bc)\,\phi - (a^3 + b^3 + c^3 - 3abc) = 0\,. \tag{4}$$

But the cubic in ϕ is

$$\phi^3 - m_2\phi^2 + m_1\phi - m = 0\,. \tag{5}$$

Therefore, equating coefficients of like powers of ϕ, and reducing,

$$a = \tfrac{1}{3}\,m_2 = \tfrac{1}{3}\,(g_1 + g_2 + g_3), \tag{6}$$

$$bc = \tfrac{1}{9}\,(m_2^2 - 3m_1) = \tfrac{1}{9}\,(g_1^2 + g_2^2 + g_3^2 - g_1g_2 - g_2g_3 - g_3g_1), \tag{7}$$

$$\begin{aligned} b^3 + c^3 &= \tfrac{1}{27}\,(27m + 2m_2^3 - 9m_1m_2) \\ &= \tfrac{1}{27}\,(2g_1^3 + 2g_2^3 + 2g_3^3 - 3g_1^2g_2 - 3g_1g_2^2 - 3g_2^2g_3 \\ &\qquad - 3g_2g_3^2 - 3g_3^2g_1 - 3g_3g_1^2 + 12g_1g_2g_3)\,. \end{aligned} \tag{8}$$

Putting $\lambda^3 = 1$ or $\lambda = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, the solution of these equations gives six possible sets of values of the form

$$a = \tfrac{1}{3}\,(g_1 + g_2 + g_3), \tag{9}$$

$$b = \tfrac{1}{3}\,(g_1 + \lambda g_2 + \lambda^2 g_3), \tag{10}$$

$$c = \tfrac{1}{3}\,(g_1 + \lambda^2 g_2 + \lambda g_3)\,. \tag{11}$$

The permutation of g_1, g_2, g_3 gives the other five sets. These six sets are all the solutions possible, since equation (7) is of the second degree and (8) of the third.

Substituting these values, we get

$$\phi = \tfrac{1}{3}\,g_1\,(1 + \iota + \iota^2) + \tfrac{1}{3}\,g_2\,(1 + \lambda\iota + \lambda^2\iota^2) + \tfrac{1}{3}\,g_3\,(1 + \lambda^2\iota + \lambda\iota^2). \tag{12}$$

Let

$$\varkappa_1 = \tfrac{1}{3}\,(1 + \iota + \iota^2), \tag{13}$$

$$\varkappa_2 = \tfrac{1}{3}\,(1 + \lambda\iota + \lambda^2\iota^2), \tag{14}$$

$$\varkappa_3 = \tfrac{1}{3}\,(1 + \lambda^2\iota + \lambda\iota^2)\,. \tag{15}$$

We derive from these immediately

$$\left.\begin{aligned} \varkappa_1^2 &= \varkappa_1, & \varkappa_1\varkappa_2 &= 0 = \varkappa_2\varkappa_1, \\ \varkappa_2^2 &= \varkappa_2, & \varkappa_1\varkappa_3 &= 0 = \varkappa_3\varkappa_1, \\ \varkappa_3^2 &= \varkappa_3, & \varkappa_2\varkappa_3 &= 0 = \varkappa_3\varkappa_2, \end{aligned}\right\} \tag{16}$$

$$\phi = g_1\varkappa_1 + g_2\varkappa_2 + g_3\varkappa_3\,. \tag{17}$$

Now if ρ_1, ρ_2, ρ_3 are the axes of ϕ, i. e. solutions of the equation $V.\rho\phi\rho = 0$, we can write ϕ in the form

$$\phi = \frac{g_1\rho_1 S.\rho_2\rho_3\,(\;) + g_2\rho_2 S.\rho_3\rho_1\,(\;) + g_3\rho_3 S.\rho_1\rho_2\,(\;)}{S.\rho_1\rho_2\rho_3}\,. \tag{18}$$

Operating by $\varkappa_1$, we get

$$\varkappa_1\phi = g_1\varkappa_1 = \frac{g_1\varkappa_1\rho_1 S\,.\,\rho_2\rho_3\,(\,) + g_2\varkappa_1\rho_2 S\rho_3\rho_1\,(\,) + g_3\varkappa_1\rho_3 S\rho_1\rho_2\,(\,)}{S\,.\,\rho_1\rho_2\rho_3}\,. \tag{19}$$

Operating by this operator on $\rho = x\rho_1 + y\rho_2 + z\rho_3$, where ρ is *any* vector, we get

$$xg_1\varkappa_1\rho_1 + yg_1\varkappa_1\rho_2 + zg_1\varkappa_1\rho_3 = xg_1\varkappa_1\rho_1 + yg_2\varkappa_1\rho_2 + zg_3\varkappa_1\rho_3\,.$$

Hence, since x, y, z are independent,

$$g_1\varkappa_1\rho_2 = g_2\varkappa_1\rho_2\,, \quad g_1\varkappa_1\rho_3 = g_3\varkappa_1\rho_3\,.$$

These equations are possible generally only if

$$\varkappa_1\rho_2 = 0\,, \quad \varkappa_1\rho_3 = 0\,,$$

$$\therefore\ \varkappa_1 = \frac{\varkappa_1\rho_1 S\,.\,\rho_2\rho_3\,(\,)}{S\,.\,\rho_1\rho_2\rho_3}\,, \tag{20}$$

$$\varkappa_2 = \frac{\varkappa_2\rho_2 S\,.\,\rho_3\rho_1\,(\,)}{S\,.\,\rho_1\rho_2\rho_3}\,, \tag{21}$$

$$\varkappa_3 = \frac{\varkappa_3\rho_3 S\,.\,\rho_1\rho_2\,(\,)}{S\,.\,\rho_1\rho_2\rho_3}\,. \tag{22}$$

Let $$\varkappa_1\rho_1 = x\rho_1 + y\rho_2 + z\rho_3\,.$$

Operating by $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ in turn we obtain

$$\begin{aligned} \varkappa_1\rho_1 &= x\varkappa_1\rho_1\,, & \therefore\ x &= 1\,. \\ y\varkappa_2\rho_2 &= 0\,, & \therefore\ y &= 0\,. \\ z\varkappa_3\rho_3 &= 0\,, & \therefore\ z &= 0\,. \end{aligned}$$

Treating each $\varkappa$ in the same manner and substituting in (20), (21), (22), we have

$$\left.\begin{aligned} \varkappa_1 &= \frac{\rho_1 S\rho_2\rho_3\,(\,)}{S\,.\,\rho_1\rho_2\rho_3}\,, \\ \varkappa_2 &= \frac{\rho_2 S\rho_3\rho_1\,(\,)}{S\,.\,\rho_1\rho_2\rho_3}\,, \\ \varkappa_3 &= \frac{\rho_3 S\rho_1\rho_2\,(\,)}{S\,.\,\rho_1\rho_2\rho_3}\,. \end{aligned}\right\} \tag{23}$$

Therefore, finally,

$$\left.\begin{aligned}\iota &= \frac{\rho_1 S\rho_2\rho_3\,(\,) + \lambda^2\rho_2 S\rho_3\rho_1\,(\,) + \lambda\rho_3 S.\rho_1\rho_2\,(\,)}{S.\rho_1\rho_2\rho_3}, \\ \iota^2 &= \frac{\rho_1 S\rho_2\rho_3\,(\,) + \lambda\rho_2 S\rho_3\rho_1\,(\,) + \lambda^2\rho_3 S.\rho_1\rho_2\,(\,)}{S.\rho_1\rho_2\rho_3}, \\ \iota^3 = 1 &= \frac{\rho_1 S\rho_2\rho_3\,(\,) + \rho_2 S\rho_3\rho_1\,(\,) + \rho_3 S.\rho_1\rho_2\,(\,)}{S.\rho_1\rho_2\rho_3}.\end{aligned}\right\} \tag{24}$$

This completes the solution, as we now have a, b, c expressed in terms of the roots and ι in terms of the axes. [We observe further that the six solutions mentioned are in the end one solution, as if we interchange g_2 and g_3 in (9–11) we likewise interchange ρ_2 and ρ_3 in (24).]

If the axes are not given, we may determine the $\varkappa$'s and ι directly from the roots and ϕ, or the numbers a, b, c and ϕ, thus:

$$\left.\begin{aligned}1 &= \varkappa_1 + \varkappa_2 + \varkappa_3, \\ \phi &= g_1\varkappa_1 + g_2\varkappa_2 + g_3\varkappa_3, \\ \phi^2 &= g_1^2\varkappa_1 + g_2^2\varkappa_2 + g_3^2\varkappa_3.\end{aligned}\right\} \tag{25}$$

Hence, solving for $\varkappa_1$, $\varkappa_2$, $\varkappa_3$,

$$\left.\begin{aligned}\varkappa_1 &= \frac{(\phi - g_2)(\phi - g_3)}{(g_1 - g_2)(g_1 - g_3)}, \\ \varkappa_2 &= \frac{(\phi - g_3)(\phi - g_1)}{(g_2 - g_3)(g_2 - g_1)}, \\ \varkappa_3 &= \frac{(\phi - g_1)(\phi - g_2)}{(g_1 - g_2)(g_1 - g_3)}.\end{aligned}\right\} \tag{26}$$

These values may be substituted in the values of ι and ι^2 in terms of $\varkappa_1$, $\varkappa_2$, $\varkappa_3$. But we may also get ι and ι^2 thus:

$$\phi - a = b\iota + c\iota^2,$$

$$\phi^2 - a^2 - 2bc = (c^2 + 2ab)\,\iota + (b^2 + 2ac)\,\iota^2,$$

hence

$$\iota = \frac{c\phi^2 - (b^2 + 2ac)\,\phi + ab^2 + ca^2 - 2bc^2}{c^3 - b^3}, \tag{27}$$

$$\iota^2 = \frac{b\phi^2 - (c^2 + 2ab)\,\phi + ac^2 + ba^2 - 2b^2c}{b^3 - c^3}, \tag{28}$$

These in turn will give $\varkappa_1, \varkappa_2, \varkappa_3$ in terms of a, b, c and ϕ.

2. If two operators are equal they must have the same a, b, c and ι.

For if
$$\left.\begin{aligned}\psi &= a' + b'\iota' + c'\iota'^2,\\ = \phi &= a + b\iota + c\iota^2,\end{aligned}\right\} \tag{29}$$
we have at once $a = a'$;
$$\therefore\ b\iota + c\iota^2 = b'\iota' + c'\iota'^2, \tag{30}$$
$$\therefore\ b\iota^2 + c = b'\iota\iota' + c'\iota\iota'^2, \tag{31}$$
$$= b'\,\frac{\rho_1 S\rho_2\rho_3\,\iota'\,(\,) + \lambda^2\rho_2 S\rho_3\rho_1\,\iota'\,(\,) + \lambda\rho_3 S\rho_1\rho_2\,\iota'\,(\,)}{S.\rho_1\rho_2\rho_3}$$
$$+ c'\,\frac{\rho_1 S\rho_2\rho_3\,\iota'^2\,(\,) + \lambda^2\rho_2 S\rho_3\rho_1\,\iota'^2\,(\,) + \lambda\rho_3 S.\rho_1\rho_2\,\iota'^2\,(\,)}{S.\rho_1\rho_2\rho_3}. \tag{32}$$
Let the axes of ψ be $\rho_1', \rho_2', \rho_3'$. Then
$$b\iota^2\rho_1' + c\rho_1' = b'\,\frac{\rho_1 S\rho_2\rho_3\rho_1'}{S\rho_1\rho_2\rho_3} + c'\,\frac{\rho_1 S\rho_2\rho_3\rho_1'}{S.\rho_1\rho_2\rho_3}. \tag{33}$$
Substituting for ι^2 its value in (24),
$$\left.\begin{aligned} b + c &= b' + c',\\ b\lambda\, S\rho_3\rho_1\rho_1' + cS\rho_3\rho_1\rho_1' &= 0,\\ b\lambda^2 S\rho_1\rho_2\rho_1' + cS\rho_1\rho_2\rho_1' &= 0,\end{aligned}\right\} \tag{34}$$
$$\therefore\ S.\rho_3\rho_1\rho_1' = 0,\quad S\rho_1\rho_2\rho_1' = 0. \tag{35}$$

Finally, we must have then
$$\left.\begin{aligned}\rho_1' &= \rho_1,\\ \text{and }\ \rho_2' &= \rho_2,\\ \rho_3' &= \rho_3,\end{aligned}\right\} \tag{36}$$
since tensors are not concerned.

Therefore
$$\iota = \iota' \text{ and } c = c',\ b = b'. \tag{37}$$
But it is to be observed that since $(\iota^2)^2 = \iota$, and $(\iota^2)^3 = 1$, we cannot, knowing that
$$\begin{aligned}\psi &= a' + b'\iota' + c'\iota'^2,\\ \phi &= a + b\iota + c\iota^2,\\ \psi &= \phi,\end{aligned}$$
assume that $b'\iota' = b\iota$, $c'\iota'^2 = c'\iota^2$; it is necessary to find the axes, and writing the forms (24) identify ι' with ι or ι^2 accordingly. (See Taber, "On Certain Identities in the Theory of Matrices," Amer. Jour., XIII, p. 165.)

3. The expression

$$bc\psi^2 + (b^3 + c^3 - 2abc)\,\psi + b^2c^2 - ab^3 - ac^3 + a^2bc$$

may be called the Hessian of the cubic in ϕ. Its factors are

$$\left(\psi + \frac{c^2 - ab}{b}\right), \quad \left(\psi + \frac{b^2 - ac}{c}\right).$$

When its axes are properly chosen it plays a part in the theory of the triangle.

The expression

$$\left(\theta + \frac{b^2 + c^2 - ab - ac}{b + c}\right)\left(\theta + \frac{\lambda b^2 + \lambda^2 c^2 - \lambda ac - \lambda^2 ab}{\lambda^2 b + \lambda c}\right)$$
$$\times\left(\theta + \frac{\lambda^2 b^2 + \lambda c^2 - \lambda^2 ac - \lambda ab}{\lambda b + \lambda^2 c}\right)$$

may be called the cubicovariant of the cubic in ϕ. It also is of importance in the theory of the triangle.

The two invariants of the cubic in ϕ are

$$\left.\begin{aligned} H &= -bc = \tfrac{1}{9}(3m_1 - m_2^2), \\ G &= -(b^3 + c^3) = \tfrac{1}{27}(9m_1m_2 - 2m_2^3 - 27m). \end{aligned}\right\} \tag{38}$$

The discriminant is

$$\Delta = (b^3 - c^3)^2 = \tfrac{1}{27}(27m^2 - 18mm_1m_2 + 4m_1^3 + 4mm_2^3 - m_1^2m_2^2). \tag{39}$$

When this is negative the roots g_1, g_2, g_3 are all real; when it is positive, two roots are imaginary; when it is zero, there are two equal roots; when $G = 0$ and $H = 0$ there are three equal roots.

4. The vector function

$$\phi_1 = \phi - a = b\iota + c\iota^2$$

has for its cubic

$$\phi_1^3 - 3bc\phi_1 - (b^3 + c^3) = 0. \tag{40}$$

Its roots are

$$\left.\begin{aligned} e_1 &= \tfrac{1}{3}(2g_1 - g_2 - g_3) = b + c, \\ e_2 &= \tfrac{1}{3}(2g_2 - g_3 - g_1) = \lambda^2 b + c\lambda, \\ e_3 &= \tfrac{1}{3}(2g_3 - g_1 - g_2) = \lambda b + \lambda^2 c, \\ & e_1 + e_2 + e_3 = 0. \end{aligned}\right\} \tag{41}$$

Hence

$$b\iota + c\iota^2 = e_1\varkappa_1 + e_2\varkappa_2 + e_3\varkappa_3. \tag{42}$$

Let

$$\wp\,.\,u$$

represent Weierstrass' $\wp$ function for the invariants

$$\left.\begin{aligned} g_2 &= 12bc = \tfrac{4}{3}(m_2^2 - 3m_1), \\ g_3 &= 4(b^3 + c^3) = \tfrac{4}{27}(27m + 2m_2^3 - 9m_1m_2). \end{aligned}\right\} \tag{43}$$

Then the periods satisfy the equations

$$\left.\begin{aligned} \wp\,.\,\omega_1 = e_1,\ \ \wp\,.\,\omega_2 = e_2,\ \ \wp\,.\,\omega_3 = e_3, \\ \omega_1 + \omega_2 + \omega_3 = 0. \end{aligned}\right\} \tag{44}$$

Now we may let

$$b\iota + c\iota^2 = \wp\,.\,\psi, \tag{45}$$

whence

$$\psi = \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3. \tag{46}$$

Further,

$$(b\iota + c\iota^2) - e_\mu = -\frac{\sigma(\psi + e_\mu)\sigma(\psi - e_\mu)}{\sigma^2\psi\,\sigma^2 e_\mu} = \left(\frac{\sigma_\mu \psi}{\sigma\psi}\right)^2$$

$$= \frac{e_\nu - e_\mu}{\operatorname{sn}^2\left(\sqrt{e_\nu - e_\mu}\,\psi,\ \sqrt{\dfrac{e_\nu - e_\mu}{e_{\nu'} - e_\mu}}\right)}. \tag{47}$$

These results, startling at first sight, are useful in the geometry of the triangle.

5. Let the transverse of ϕ be ϕ'. We have at once, since the roots are the same, that ϕ' has the same a, b, c as ϕ. But ι becomes

$$\iota' = \frac{V\rho_2\rho_3 S.\rho_1(\,) + \lambda^2 V\rho_3\rho_1 S.\rho_2(\,) + \lambda V\rho_1\rho_2 S\rho_3(\,)}{S.\rho_1\rho_2\rho_3}. \tag{48}$$

We have easily

$$\phi - \phi' = 2V.\varepsilon(\,) = \frac{g_1\rho_1 S.V\rho_2\rho_3(\,) - g_1 V\rho_2\rho_3 S\rho_1(\,) + \ldots.}{S.\rho_1\rho_2\rho_3} \tag{49}$$

$$= \frac{g_1 V.(V\rho_1 V\rho_2\rho_3)(\,) + g_2 V.(V.\rho_2 V\rho_3\rho_1)(\,) + g_3 V.(V.\rho_3 V\rho_1\rho_2)(\,)}{S.\rho_1\rho_2\rho_3}, \tag{50}$$

$$\therefore\ 2\varepsilon = \frac{g_1 V\rho_1 V\rho_2\rho_3 + g_2 V\rho_2 V\rho_3\rho_1 + g_3 V\rho_3 V\rho_1\rho_2}{S.\rho_1\rho_2\rho_3} \tag{51}$$

$$= \frac{(g_2 - g_3)\rho_1 S\rho_2\rho_3 + (g_3 - g_1)\rho_2 S.\rho_3\rho_1 + (g_1 - g_2)\rho_3 S\rho_1\rho_2}{S.\rho_1\rho_2\rho_3} \tag{52}$$

$$= \frac{(e_2 - e_3)\rho_1 S.\rho_2\rho_3 + (e_3 - e_1)\rho_2 S\rho_3\rho_1 + (e_1 - e_2)\rho_3 S\rho_1\rho_2}{S.\rho_1\rho_2\rho_3}. \tag{53}$$

6. Let the tensor of ϕ be defined as the arithmetical cube root of m, i. e.

$$T.\phi = \sqrt[3]{[a^3 + b^3 + c^3 - 3abc]} = \sqrt[3]{m} = \sqrt[3]{g_1 g_2 g_3}. \tag{54}$$

Let the versor of ϕ be $U.\phi$,

$$U.\phi = \frac{a}{T\phi} + \frac{b}{T\phi}\iota + \frac{c}{T\phi}\iota^2 = \frac{\phi}{T\phi} \tag{55}$$

$$= \frac{g_1}{\sqrt[3]{g_1g_2g_3}}\varkappa_1 + \frac{g_2}{\sqrt[3]{g_1g_2g_3}}\varkappa_2 + \frac{g_3}{\sqrt[3]{g_1g_2g_3}}\varkappa_3. \tag{56}$$

Let

$$\left.\begin{aligned} \frac{g_1}{\sqrt[3]{g_1g_2g_3}} &= e^{\theta+\eta}, \\ \frac{g_2}{\sqrt[3]{g_1g_2g_3}} &= e^{\lambda\theta+\lambda^2\eta}, \\ \frac{g_3}{\sqrt[3]{g_1g_2g_3}} &= e^{\lambda^2\theta+\lambda\eta}, \end{aligned}\right\} \tag{57}$$

whence

$$\left.\begin{aligned} \therefore\ \frac{a}{T.\phi} &= \tfrac{1}{3}\left[e^{\theta+\eta} + e^{\lambda\theta+\lambda^2\eta} + e^{\lambda^2\theta+\lambda\eta}\right] = f_0(\theta, \eta), \\ \frac{b}{T.\phi} &= \tfrac{1}{3}\left[e^{\theta+\eta} + \lambda e^{\lambda\theta+\lambda^2\eta} + \lambda^2 e^{\lambda^2\theta+\lambda\eta}\right] = f_1(\theta, \eta), \\ \frac{c}{T.\phi} &= \tfrac{1}{3}\left[e^{\theta+\eta} + \lambda^2 e^{\lambda\theta+\lambda^2\eta} + \lambda e^{\lambda^2\theta+\lambda\eta}\right] = f_2(\theta, \eta), \end{aligned}\right\} \tag{58}$$

(compare Taber, Amer. Jour., XIII, p. 169 et seq.)

$$\therefore\quad \phi = T\phi\left[f_0(\theta, \eta) + \iota f_1(\theta, \eta) + \iota^2 f_2(\theta, \eta)\right]. \tag{59}$$

If

$$\left.\begin{aligned} f_0\theta &= \tfrac{1}{3}(e^{\theta} + e^{\lambda\theta} + e^{\lambda^2\theta}), \\ f_1\theta &= \tfrac{1}{3}(e^{\theta} + \lambda e^{\lambda\theta} + \lambda^2 e^{\lambda^2\theta}), \\ f_2\theta &= \tfrac{1}{3}(e^{\theta} + \lambda^2 e^{\lambda\theta} + \lambda e^{\lambda^2\theta}), \end{aligned}\right\} \tag{60}$$

$$\phi = T\phi\,.\left[f_0\theta + \iota f_1\theta + \iota^2 f_2\theta\right]\left[f_0\eta + \iota f_2\eta + \iota^2 f_1\eta\right]. \tag{61}$$

We see easily

$$\left.\begin{aligned} f_0\theta &= L_{n=\infty}\left\{1 + \frac{\theta^3}{3!} + \frac{\theta^6}{6!} + \ldots \frac{\theta^{3n}}{(3n)!}\right\}, \\ f_1\theta &= L_{n=\infty}\left\{\frac{\theta^2}{2!} + \frac{\theta^5}{5!} + \frac{\theta^8}{8!} + \ldots \frac{\theta^{3n-1}}{(3n-1)!}\right\}, \\ f_2\theta &= L_{n=\infty}\left\{\frac{\theta}{1!} + \frac{\theta^4}{4!} + \frac{\theta^7}{7!} + \ldots \frac{\theta^{3n-2}}{(3n-2)!}\right\}, \end{aligned}\right\} \tag{62}$$

and

$$\left.\begin{aligned} f_0\theta &= \tfrac{1}{3}\left(e^{\theta} + 2e^{-\frac{1}{2}\theta}\cos.\frac{\theta\sqrt{3}}{2}\right) \quad, \\ f_1\theta &= \tfrac{1}{3}\left(e^{\theta} - e^{-\frac{1}{2}\theta}\cos\frac{\theta\sqrt{3}}{2} - \sqrt{3}\,.\,e^{-\frac{1}{2}\theta}\sin\frac{\theta\sqrt{3}}{2}\right), \\ f_2\theta &= \tfrac{1}{3}\left(e^{\theta} - e^{-\frac{1}{2}\theta}\cos\frac{\theta\sqrt{3}}{2} + \sqrt{3}\,.\,e^{-\frac{1}{2}\theta}\sin\frac{\theta\sqrt{3}}{2}\right). \end{aligned}\right\} \tag{63}$$

Also

$$\left.\begin{aligned} \theta + \eta &= \tfrac{2}{3}\lg g_1 - \tfrac{1}{3}\lg g_2 - \tfrac{1}{3}\lg g_3, \\ \lambda\theta + \lambda^2\eta &= -\tfrac{1}{3}\lg g_1 + \tfrac{2}{3}\lg g_2 - \tfrac{1}{3}\lg g_3, \end{aligned}\right\}. \tag{64}$$

$$\therefore\ \left.\begin{aligned} \theta &= \tfrac{1}{3}\lg g_1 + \tfrac{1}{3}\lambda^2\lg g_2 + \tfrac{1}{3}\lambda\ \lg g_3, \\ \eta &= \tfrac{1}{3}\lg g_1 + \tfrac{1}{3}\lambda\ \lg g_2 + \tfrac{1}{3}\lambda^2\lg g_3, \end{aligned}\right\} \tag{65}$$

$$\left.\begin{aligned} \theta &= \tfrac{1}{3}\lg\left(g_1 g_2^{-\frac{1}{2}} g_3^{-\frac{1}{2}}\right) + \frac{\sqrt{-1}}{\sqrt{3}}\lg\left(g_2^{-\frac{1}{2}} g_3^{+\frac{1}{2}}\right), \\ \eta &= \tfrac{1}{3}\lg\left(g_1 g_2^{-\frac{1}{2}} g_3^{-\frac{1}{2}}\right) - \frac{\sqrt{-1}}{\sqrt{3}}\lg\left(g_2^{-\frac{1}{2}} g_3^{\frac{1}{2}}\right)\ , \end{aligned}\right\} \tag{66}$$

$$\left.\begin{aligned} \theta + \eta &= \tfrac{1}{3}\lg.\frac{g_1^3}{m}, \\ \theta - \eta &= \frac{\sqrt{-1}}{\sqrt{3}}\lg.\frac{g_3}{g_2}. \end{aligned}\right\} \tag{67}$$

7. Returning now to the solution given in §1, it was stated that the roots g_1, g_2, g_3 might be permuted, giving six different solutions of the equations there given. These six solutions suggest six related functions which I designate as conjugates and co-functions. Of course a is the same for all.

The first of these six is derived by interchanging g_2 and g_3 (but leaving the axes unchanged); whence, since this interchanges b and c,

$$\text{Co.}\,\phi = a + c\iota + b\iota^2. \tag{68}$$

We have

$$\phi + \text{Co.}\,\phi = 2a + (b + c)(\iota + \iota^2). \tag{69}$$

In this expression $\iota + \iota^2$ is an operator whose form is

$$\iota + \iota^2 = \frac{2\rho_1 S\rho_2\rho_3\,(\ ) - \rho_2 S\rho_3\rho_1\,(\ ) - \rho_3 S\rho_1\rho_2\,(\ )}{S.\rho_1\rho_2\rho_3}. \tag{70}$$

Hence its roots are 2, -1, -1, and axes ρ_1, ρ_2, ρ_3. It is of much importance in the theory of the triangle.

$$\phi - \text{Co.}\,\phi = (b - c)(\iota - \iota^2), \tag{71}$$

wherein
$$\iota - \iota^2 = \sqrt{-3}\,\frac{\rho_3 S\rho_1\rho_2\,(\,) - \rho_2 S\,.\,\rho_3\rho_1\,(\,)}{S\,.\,\rho_1\rho_2\rho_3}, \tag{72}$$
$$= -\sqrt{-3}\,\frac{(\rho_3 S\rho_2 + \rho_2 S\rho_3)\,V.\,\rho_1\,(\,)}{S\,.\,\rho_1\rho_2\rho_3}. \tag{73}$$

Hence its roots are $\pm\sqrt{-3}$, and axes ρ_3 and ρ_2. Having but two axes its nullity is one, hence vacuity must also be at least one, and since the extension annulled is that along ρ_1, which is not an axis, its vacuity is just one. (Taber, Amer. Jour., XII, p. 365.)

When
$$\phi = \text{Co.}\,\phi, \quad b = c \text{ and } g_2 = g_3. \tag{74}$$
Obviously
$$\text{Co.}\,\phi = T\phi\,[f_0\theta + f_1\theta\,.\,\iota^2 + f_2\theta\,.\,\iota][f_0\eta + \iota f_1\eta + \iota^2 f_2\eta]. \tag{75}$$

8. The next of the six is got by writing g_3, g_1, g_2 for g_1, g_2, g_3 respectively. It is the first conjugate of ϕ:

$$\varkappa\,.\,\phi = a + \lambda b\iota + \lambda^2 c\iota^2, \tag{76}$$
$$= g_3\varkappa_1 + g_1\varkappa_2 + g_2\varkappa_3. \tag{77}$$

The second conjugate is $\varkappa\,.\,\varkappa\phi = \varkappa^2\phi$,

$$\varkappa^2.\,\phi = a + \lambda^2 b\iota + \lambda c\iota^2, \tag{78}$$
$$= g_2\varkappa_1 + g_3\varkappa_2 + g_1\varkappa_3. \tag{79}$$

The properties of these two are indicated by Taber, Amer. Jour., XIII, as also of the parts of each, of ϕ, $\varkappa\phi$, $\varkappa^2\phi$, viz. $S\phi$, $V_1\phi$, $V_2\phi$.

9. The co-functions of the conjugates have of course the same relation to the conjugates as the co-function of ϕ has to ϕ. Thus

$$\text{Co.}\,\varkappa\,.\,\phi = a + \lambda c\iota + \lambda^2 b\iota^2 \tag{80}$$
$$= \varkappa\,.\,\text{Co.}\,\phi, \tag{81}$$
$$\text{Co.}\,\varkappa^2.\,\phi = \varkappa^2.\,\text{Co.}\,\phi. \tag{82}$$

The six operators
$$\phi,\ \text{Co.}\,\phi,\ \varkappa\phi,\ \varkappa\,.\,\text{Co.}\,\phi,\ \varkappa^2\phi,\ \varkappa^2.\,\text{Co.}\,\phi$$

are the six related operators referred to in §7.

10. Various combinations of these produce coefficients of the forms

$$a + b + c \;,$$
$$a + \lambda b + \lambda^2 c,$$
$$a + \lambda^2 b + \lambda c \;;$$
$$a^2 + b^2 + c^2, \;\; a^2 + \lambda b^2 + \lambda^2 c^2, \;\; a^2 + \lambda^2 b^2 + \lambda c^2;$$

etc., which might be called *triskew* symmetric.

II.—*The Nonion Form of* ϕ.

1. Let there be any three vectors chosen, not coplanar, say α, β, γ. Then we may write

$$\phi\alpha = \frac{\alpha S\beta\gamma\phi\alpha + \beta S\gamma\alpha\phi\alpha + \gamma S\alpha\beta\phi\alpha}{S\alpha\beta\gamma}, \tag{1}$$

etc. If we write

$$\phi\alpha = \alpha_1, \;\; \phi\beta = \beta_1, \;\; \phi\gamma = \gamma_1, \tag{2}$$

since any vector ρ may be expressed in terms of α, β, γ, we have in the most general case

$$\begin{aligned}\phi = &\frac{\alpha S\beta\gamma\alpha_1 S.\beta\gamma\,(\;)}{S.\alpha\beta\gamma} + \frac{\beta S\gamma\alpha\alpha_1 S.\beta\gamma\,(\;)}{S.\alpha\beta\gamma} + \frac{\gamma S\alpha\beta\alpha_1 S.\beta\gamma\,(\;)}{S.\alpha\beta\gamma} \\ &+ \frac{\alpha S\beta\gamma\beta_1 S.\gamma\alpha\,(\;)}{S.\alpha\beta\gamma} + \frac{\beta S\gamma\alpha\beta_1 S.\gamma\alpha\,(\;)}{S.\alpha\beta\gamma} + \frac{\gamma S\alpha\beta\beta_1 S.\gamma\alpha\,(\;)}{S.\alpha\beta\gamma} \\ &+ \frac{\alpha S\beta\gamma\gamma_1 S.\alpha\beta\,(\;)}{S.\alpha\beta\gamma} + \frac{\beta S\gamma\alpha\gamma_1 S.\alpha\beta\,(\;)}{S.\alpha\beta\gamma} + \frac{\gamma S\alpha\beta\gamma_1 S.\alpha\beta\,(\;)}{S.\alpha\beta\gamma}, \end{aligned} \tag{3}$$

$$\begin{aligned}&= \nu_1' S.\beta\gamma\alpha_1 + \nu_1'' S.\gamma\alpha\alpha_1 + \nu_1''' S.\alpha\beta\alpha_1 \\ &+ \nu_2' S.\beta\gamma\beta_1 + \nu_2'' S.\gamma\alpha\beta_1 + \nu_2''' S.\alpha\beta\beta_1 \\ &+ \nu_3' S.\beta\gamma\gamma_1 + \nu_3'' S.\gamma\alpha\gamma_1 + \nu_3''' S.\alpha\beta\gamma_1, \end{aligned} \tag{4}$$

wherein the form of the ν's is evident by comparison of (3) and (4).

The operators ν are such that

$$\nu_1' + \nu_2'' + \nu_3''' = 1, \tag{5}$$

$$\left.\begin{aligned} \nu_r^{(r)}\nu_r^{(r)} &= \nu_r^{(r)}, \\ \nu_r^{(s)}\nu_t^{(r)} &= \nu_t^{(s)}, \\ \nu_r^{(s)}\nu_y^{(t)} &= 0 \;. \end{aligned}\right\} \tag{6}$$

They have the following multiplication-table:

	ν_1'	ν_1''	ν_1'''	ν_2'	ν_2''	ν_2'''	ν_3'	ν_3''	ν_3'''
ν_1'	ν_1'	0	0	ν_2'	0	0	ν_3'	0	0
ν_1''	ν_1''	0	0	ν_2''	0	0	ν_3''	0	0
ν_1'''	ν_1'''	0	0	ν_2'''	0	0	ν_3'''	0	0
ν_2'	0	ν_1'	0	0	ν_2'	0	0	ν_3'	0
ν_2''	0	ν_1''	0	0	ν_2''	0	0	ν_3''	0
ν_2'''	0	ν_1'''	0	0	ν_2'''	0	0	ν_3'''	0
ν_3'	0	0	ν_1'	0	0	ν_2'	0	0	ν_3'
ν_3''	0	0	ν_1''	0	0	ν_2''	0	0	ν_3''
ν_3'''	0	0	ν_1'''	0	0	ν_2'''	0	0	ν_3'''

2. Of these nine operators

$$\nu_1', \ \nu_2'', \ \nu_3'''$$

have each a cubic of the form

$$\nu^3 - \nu^2 = 0. \tag{7}$$

Both nullity and vacuity are two.

The remaining six have, since they each annul extension in two directions, a nullity two; and, since the axis is included in the extension annulled, vacuity three. The cubic is

$$\nu^3 = 0. \tag{8}$$

3. When α, β, γ coincide with the axes, all the terms of ϕ vanish except the three involving

$$\nu_1', \ \nu_2'', \ \nu_3'''.$$

The roots are in this case

$$S.\beta\gamma\alpha_1, \quad S.\gamma\alpha\beta_1, \quad S.\alpha\beta\gamma_1.$$

4. Let us write

$$\left.\begin{aligned}
\nu_0 &= \nu_1' + \nu_2'' + \nu_3''', & \nu_3 &= \nu_1'' + \nu_2''' + \nu_3', & \nu_6 &= \nu_1''' + \nu_2' + \nu_3'', \\
\nu_1 &= \nu_1' + \lambda^2\nu_2'' + \lambda\nu_3''', & \nu_4 &= \nu_1'' + \lambda^2\nu_2''' + \lambda\nu_3', & \nu_7 &= \nu_1''' + \lambda^2\nu_2' + \lambda\nu_3'', \\
\nu_2 &= \nu_1' + \lambda\nu_2'' + \lambda^2\nu_3''', & \nu_5 &= \nu'' + \lambda\nu_2''' + \lambda^2\nu_3', & \nu_8 &= \nu_1''' + \lambda\nu_2' + \lambda\nu_3''.
\end{aligned}\right\} \tag{9}$$

Then it is easily verified that *these* ν's have the multiplication-table:

	ν_0	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	ν_8
ν_0	ν_0	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	ν_8
ν_1	ν_1	ν_2	ν_0	$\lambda^2\nu_4$	$\lambda^2\nu_5$	$\lambda^2\nu_3$	$\lambda\nu_7$	$\lambda\nu_8$	$\lambda\nu_6$
ν_2	ν_2	ν_0	ν_1	$\lambda\nu_5$	$\lambda\nu_3$	$\lambda\nu_4$	$\lambda^2\nu_8$	$\lambda^2\nu_6$	$\lambda^2\nu_7$
ν_3	ν_3	ν_4	ν_5	ν_6	ν_7	ν_8	ν_0	ν_1	ν_2
ν_4	ν_4	ν_5	ν_3	$\lambda^2\nu_7$	$\lambda^2\nu_8$	$\lambda^2\nu_6$	$\lambda\nu_1$	$\lambda\nu_2$	$\lambda\nu_0$
ν_5	ν_5	ν_3	ν_4	$\lambda\nu_8$	$\lambda\nu_6$	$\lambda\nu_7$	$\lambda^2\nu_2$	$\lambda^2\nu_0$	$\lambda^2\nu_1$
ν_6	ν_6	ν_7	ν_8	ν_0	ν_1	ν_2	ν_3	ν_4	ν_5
ν_7	ν_7	ν_8	ν_6	$\lambda^2\nu_1$	$\lambda^2\nu_2$	$\lambda^2\nu_0$	$\lambda\nu_4$	$\lambda\nu_5$	$\lambda\nu_3$
ν_8	ν_8	ν_6	ν_7	$\lambda\nu_2$	$\lambda\nu_0$	$\lambda\nu_1$	$\lambda^2\nu_5$	$\lambda^2\nu_3$	$\lambda^2\nu_4$

For each we have $\nu^3 = 1$. Also

$$\left.\begin{array}{c} \nu_3\nu_4 = \lambda\nu_4\nu_3 \\ \nu_4 = \lambda\nu_1\nu_3,\ \nu_5 = \lambda^2\nu_1^2\nu_3,\ \nu_6 = \nu_3^2,\ \nu_7 = \lambda^2\nu_1\nu_3^2,\ \ \nu_8 = \lambda\nu_1^2\nu_3^2, \\ \nu_0 = 1,\qquad \nu_3\nu_1 = \lambda\nu_1\nu_3,\quad \nu_3^2\nu_1 = \lambda^2\nu_1\nu_3^2, \\ \nu_3\nu_1^2 = \lambda^2\nu_1^2\nu_3,\quad \nu_3^2\nu_1^2 = \lambda\nu_1^2\nu_3^2. \end{array}\right\} \tag{10}$$

5. Since each of these vids satisfies the equation

$$\nu^3 - 1 = 0, \tag{11}$$

it is a vid of the form ι of the first part of this paper. Writing them out in full, we have

$$\left.\begin{aligned}
\nu_0 &= \frac{\alpha S\beta\gamma\,(\,) + \beta S\gamma\alpha\,(\,) + \gamma S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_1 &= \frac{\alpha S\beta\gamma\,(\,) + \lambda^2\beta S\gamma\alpha\,(\,) + \lambda\gamma S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_2 &= \frac{\alpha S\beta\gamma\,(\,) + \lambda\beta S\gamma\alpha\,(\,) + \lambda^2\gamma S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_3 &= \frac{\beta S\beta\gamma\,(\,) + \gamma S\gamma\alpha\,(\,) + \alpha S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_4 &= \frac{\beta S\beta\gamma\,(\,) + \lambda^2\gamma S\gamma\alpha\,(\,) + \lambda\alpha S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_5 &= \frac{\beta S\beta\gamma\,(\,) + \lambda\gamma S\gamma\alpha\,(\,) + \lambda^2\alpha S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_6 &= \frac{\gamma S.\beta\gamma\,(\,) + \alpha S\gamma\alpha\,(\,) + \beta S.\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_7 &= \frac{\gamma S\beta\gamma\,(\,) + \lambda^2\alpha S\gamma\alpha\,(\,) + \lambda\beta S\alpha\beta\,(\,)}{S.\alpha\beta\gamma}, \\
\nu_8 &= \frac{\gamma S\beta\gamma\,(\,) + \lambda\alpha S\gamma\alpha\,(\,) + \lambda^2\beta S.\alpha\beta\,(\,)}{S.\alpha\beta\gamma},
\end{aligned}\right\} \tag{12}$$

The axes and corresponding roots are:

$$\left.\begin{array}{lllll}
1 = \nu_0: & \alpha \; , & \beta \; , & \gamma \; ; & 1,\; 1,\; 1\; ; \\
\nu_1: & \alpha \; , & \beta \; , & \gamma \; ; & 1,\; \lambda^2,\; \lambda\; ; \\
\nu_2: & \alpha \; , & \beta \; , & \gamma \; ; & 1,\; \lambda,\; \lambda^2 ; \\
\nu_3: & \alpha + \beta + \gamma, & \alpha + \lambda\beta + \lambda^2\gamma, & \alpha + \lambda^2\beta + \lambda\gamma ; & 1,\; \lambda^2,\; \lambda\; ; \\
\nu_4: & \alpha + \beta + \lambda^2\gamma, & \lambda^2\alpha + \beta + \gamma, & \alpha + \lambda^2\beta + \gamma ; & 1,\; \lambda^2,\; \lambda\; ; \\
\nu_5: & \lambda^2\alpha + \lambda^2\beta + \gamma, & \lambda^2\alpha + \beta + \lambda^2\gamma, & \alpha + \lambda^2\beta + \lambda^2\gamma ; & 1,\; \lambda^2,\; \lambda\; ; \\
\nu_6: & \alpha + \beta + \gamma, & \alpha + \lambda\beta + \lambda^2\gamma, & \alpha + \lambda^2\beta + \lambda\gamma ; & 1,\; \lambda,\; \lambda^2 ; \\
\nu_7: & \lambda^2\alpha + \lambda^2\beta + \gamma, & \lambda^2\alpha + \beta + \lambda^2\gamma, & \alpha + \lambda^2\beta + \lambda^2\gamma ; & \lambda^2,\; 1,\; \lambda\; ; \\
\nu_8: & \alpha + \beta + \lambda^2\gamma, & \lambda^2\alpha + \beta + \gamma, & \alpha + \lambda^2\beta + \gamma ; & \lambda,\; \lambda^2,\; 1\,.
\end{array}\right\} \quad (13)$$

6. Since α, β, γ may be taken in an infinity of ways, any vector operator ϕ may be broken up into nine elements in an infinite number of ways.

Since there are but nine nonion cube roots of unity, or nine solutions of the equation $\phi^3 = 1$, they are all expressed above, *for any particular set of vectors* (α, β, γ), with their axes and roots.

7. If the axes of ϕ are α, β, γ, then

$$\phi = a_0 + a_1\nu_1 + a_2\nu_2. \quad (14)$$

Any other operator may be reduced to this nonion form in terms of α, β, γ, giving

$$\psi = a_0' + \sum_1^8 {}_r a_r'\nu_r. \quad (15)$$

Since the ν's are all unit operators, it is plain that

$$S.\psi = a_0'. \quad (16)$$

Again, if $\psi = \chi$, then the coefficients of any one of the ν's must be identically equal. For we can find a ν, complementary to the one considered, whose product into it will give either ν_0, $\lambda\nu_0$ or $\lambda^2\nu_0$. Then if we multiply each side by this operator and take scalars, we have at once the result stated.

8. We have

$$\left.\begin{aligned}
\phi\psi = {} & a_0a_0' + a_1a_2' + a_2a_1' + \nu_1(a_0a_1' + a_1a_0' + a_2a_2') + \nu_2(a_0a_2' + a_1a_1' + a_2a_0') \\
& + \nu_3(a_0a_3' + \lambda^2a_1a_5' + \lambda a_2a_4') + \nu_4(a_0a_4' + \lambda^2a_1a_3' + \lambda a_2a_5') \\
& + \nu_5(a_0a_5' + \lambda^2a_1a_4' + \lambda a_2a_3') + \nu_6(a_0a_6' + \lambda a_1a_8' + \lambda^2a_2a_7') \\
& + \nu_7(a_0a_7' + \lambda a_1a_6' + \lambda^2a_2a_8') + \nu_8(a_0a_8' + \lambda a_1a_7' + \lambda^2a_2a_6'),
\end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned}\psi\phi = a_0a_0' + a_1a_2' + a_2a_1' + \nu_1(a_0a_1' + a_1a_0' + a_2a_2') + \nu_2(a_0a_2' + a_1a_1' + a_2a_0')\\ + \nu_3(a_0a_3' + a_1a_5' + a_2a_4') + \nu_4(a_0a_4' + a_1a_3' + a_2a_5') + \nu_5(a_0a_5' + a_1a_4' + a_2a_3')\\ + \nu_6(a_0a_6' + a_1a_8' + a_2a_7') + \nu_7(a_0a_7' + a_1a_6' + a_2a_8') + \nu_8(a_0a_8' + a_1a_7' + a_2a_6').\end{aligned}\right\} \quad (18)$$

It is easily apparent now that always

$$S.\phi\psi = S.\psi\phi, \quad S.\nu_1\phi\psi = S.\nu_1\psi\phi, \quad S.\nu_2\phi\psi = S.\nu_2\psi\phi. \quad (19)$$

Also

$$\begin{aligned}\phi\psi - \psi\phi = (\lambda - 1)\{\nu_3(-\lambda^2a_1a_5' + a_2a_4') + \nu_4(-\lambda^2a_1a_3' + a_2a_5')\\ + \nu_5(-\lambda^2a_1a_4' + a_2a_3') + \nu_6(a_1a_8' - \lambda^2a_2a_7') + \nu_7(a_1a_6' - \lambda^2a_2a_8')\\ + \nu_8(a_1a_7' - \lambda^2a_2a_6')\}. \quad (20)\end{aligned}$$

We deduce from this that $\phi\psi = \psi\phi$ only when each of these coefficients vanishes; that is, when either—

First:

$$\frac{a_1}{a_2} = \frac{\lambda a_3'}{a_4'} = \frac{\lambda a_4'}{a_5'} = \frac{\lambda a_5'}{a_3'} = \frac{\lambda^2 a_6'}{a_7'} = \frac{\lambda^2 a_7'}{a_8'} = \frac{\lambda^2 a_8'}{a_6'}, \quad (21)$$

i. e. when

$$\frac{a_3'}{a_4'} = \frac{a_4'}{a_5'} = \frac{a_5'}{a_3'},$$

whence

$$\left.\begin{array}{ll} a_4' = \lambda^m a_3', & \\ a_5' = \lambda^{2m}a_3', & a_1 = \lambda^{2m}a_2, \\ a_7' = \lambda^m a_6', & m = 1 \text{ or } 2 \text{ or } 3, \\ a_8' = \lambda^{2m}a_6', & \end{array}\right\} \quad (22)$$

or *Second*, when $a_1 : a_2 \neq \lambda^{2m}$, then a_3', a_4', a_5', a_6', a_7', a_8' are all zero.

[a_1, a_2, etc., may be imaginary, or complex, whence we cannot break these up farther.]

In the second case ϕ and ψ are clearly *coaxal*. In the first case, if

$$\left.\begin{array}{llll} m = 3 \text{ and } & a_1 = a_2, & g_1 + \lambda^2g_2 + \lambda g_3 = g_1 + \lambda g_2 + \lambda^2 g_3, & \therefore g_2 = g_3; \\ m = 1, & a_1 = \lambda a_2, & g_1 + \lambda^2g_2 + \lambda g_3 = \lambda g_1 + \lambda^2 g_2 + g_3, & \therefore g_1 = g_3; \\ m = 2, & a_1 = \lambda^2 a_2, & g_1 + \lambda^2g_2 + \lambda g_3 = \lambda^2 g_1 + g_2 + \lambda g_3, & \therefore g_1 = g_2; \end{array}\right\} \quad (23)$$

hence in either case, ϕ has two equal roots.

Substituting the values of $a_0' \ldots . a_8'$ in (15) and reducing by (12),

$$\psi = \frac{(a_0' + a_1' + a_2')\,\alpha S\beta\gamma(\,)}{S\alpha\beta\gamma} + \frac{(a_0' + \lambda^2a_1' + \lambda a_2')\,\beta S\gamma\alpha(\,)}{S\alpha\beta\gamma} + \frac{(a_0' + \lambda a_1' + \lambda^2 a_2')\,\gamma S\alpha\beta(\,)}{S.\alpha\beta\gamma}$$

$$+ \left.\begin{array}{l} \dfrac{3a_3'\beta S\beta\gamma(\,)}{S.\alpha\beta\gamma} \\ + \dfrac{3a_6'\gamma S\beta\gamma(\,)}{S.\alpha\beta\gamma} \end{array}\right\} \text{or} + \left.\begin{array}{l} \dfrac{3a_3'\gamma S\gamma\alpha(\,)}{S.\alpha\beta\gamma} \\ + \dfrac{3a_6'\alpha S\gamma\alpha(\,)}{S.\alpha\beta\gamma} \end{array}\right\} \text{or} + \left.\begin{array}{l} \dfrac{3a_3'\alpha S\alpha\beta(\,)}{S.\alpha\beta\gamma} \\ + \dfrac{3a_6'\beta S\alpha\beta}{S.\alpha\beta\gamma} \end{array}\right\} \quad (24)$$

Therefore, ψ must have for two of its axes, two directions that are taken in the plane of the axes of ϕ corresponding to the equal roots of ϕ. Further, since we might have expressed ϕ in terms of ψ, and similar conclusions would result, hence ψ must have equal roots, and we have finally:

Two operators ϕ, ψ are commutative only when (1) they have the same axes; (2) each has a pair of equal roots, the planes of the two pairs being one and the same plane. The product will be of the same nature as the factors.*

As a corollary, ϕ and ϕ' are never commutative unless $\phi = \phi'$.

In this case $\alpha \perp \beta \perp \gamma$ and the nine unit nonions become much simpler.

9. When ϕ and ψ are commutative, then

$$\psi = a_0' + a_1'\nu_1 + a_2'\nu_2 + a_3'(\nu_3 + \lambda^m\nu_4 + \lambda^{2m}\nu_5) + a_6'(\nu_6 + \lambda^m\nu_7 + \lambda^{2m}\nu_8), \quad (23)$$

in which a_3' or a_6' or both may be zero.

10. There is a large field remaining here which is of great use in any application of Quaternions, aside from its importance as a part of the algebra Nonions.

ILLINOIS COLLEGE, *August* 1, 1895.

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* For a fuller discussion see paper read before the American Mathematical Society in August, 1896.